Chin. J. Astron. Astrophys. Vol. 5 (2005), **Suppl.**, 258–262
(*http://www.chjaa.org*)

Chinese Journal of
Astronomy and
Astrophysics

Radial Epicyclic Frequency in a Perfect Fluid Disk Around Compact Objects

Lei Qian ⋆ and Xue-Bing Wu

Department of Astronomy, Peking University, Beijing 100871

Abstract The widely used expression of the radial epicyclic frequency, which is important in the relativistic precession model to interpret the observed quasi-periodic oscillation(QPO) in X-ray binaries, is derived by considering the perturbation of a circular-orbiting test particle around a compact object. However, this is not consistent with the real configuration of an accretion disk. Here we obtain the expressions of the radial epicyclic frequency of a perfect fluid disk in both Schwarzschild and Kerr metrics and find that they are different from the previous expressions if the pressure of fluid is considered. The effects of these new expressions on the theoretical QPO models are discussed.

Key words: accretion: accretion disk — black hole physics — relativity — X-ray: binaries

1 INTRODUCTION

Since the first observation of kHz QPOs in February 1996, several theoretical models have been advanced to account for this phenomenon (Van der Klis 2000). Among them, the relativistic precession model can interpret the phenomenon well. This model depends on three characteristic frequencies, namely the orbital frequency ($\Omega \equiv 2\pi\nu_{\rm orb}$), the radial epicyclic frequency ($\kappa \equiv 2\pi\nu_r$), and the vertical epicyclic frequency ($\zeta \equiv 2\pi\nu_{\rm vert}$). Usually, the amplitude of the vertical epicyclic oscillation is much smaller than the radial one (Kluźniak et al. 2004), therefore we here focus on the radial epicyclic frequency only.

The radial epicyclic frequency in a Kerr space-time was first derived in 1987 based on the consideration of the perturbation of circular-orbiting test particle (Okazaki, Kato & Fukue 1987). Such a frequency in a Schwarzschild space-time has been used to interpret the observed QPOs in neutron star X-ray binaries (Stella & Vietri 1999). The mass of the neutron star needed to fit the observations is around 2.0 $M_\odot$, which is, however, significantly larger than the observed value of neutron star mass (1.4 $M_\odot$). There may be many factors that account for this inconsistency. For example, the expression of the radial epicyclic frequency may be one of them, since it was previously derived for a test particle. An exact expression of the radial epicyclic frequency in a fluid disk should be derived. This has been done, recently for a perfect fluid disk in a Schwarzschild space-time (Rezzolla et al. 2003). Here we extend this study to the case in a Kerr space-time.

⋆ E-mail: qianlivan@pku.edu.cn

In this paper, we first briefly review some previous works. Then we derive our expression on the radial epicyclic frequency in a Kerr space-time. Finally we give some discussions on our result and its relation to QPO phenomena.

2 RADIAL EPICYCLIC FREQUENCY IN THE CIRCULAR-ORBITING TEST PARTICLE SCENARIO

Let's consider a test particle moving in a circular orbit in a Kerr space-time. The four-velocity of this particle would be

$$u^{\alpha} = u^{t}(1, 0, \Omega, 0) \ , \tag{1}$$

This particle obeys the conservation laws of energy and momentum. The equation of motion is decided by the conservation equation of the energy-momentum tensor

$$T^{\alpha\beta}{}_{;\alpha} = 0 \ , \tag{2}$$

where $T^{\alpha\beta} = u^{\alpha}u^{\beta}$. However, in this special case the particle moves along the geodesic. Using this principle, the equation of motion can be simplified. Finally, the equation of motion can take the form of

$$(v^{\beta}{}_{;\alpha} + v^{\beta}\partial_{\alpha}\ln u^{t})v^{\alpha} = 0 \ , \tag{3}$$

where $v^{\alpha} = \frac{u^{\alpha}}{u^{t}}$ is the physical velocity of the particle. Assuming the perturbed velocity is of the form

$$v^{\alpha} = (1, \delta v^{r}, \Omega + \delta v^{\phi}, 0) \ , \tag{4}$$

and substituting it into Eq. (3), after preserving only the first order terms we obtain a set of perturbation equations. From them we can get the square of the radial epicyclic frequency as

$$\kappa^{2} = -(\frac{\partial\Omega}{\partial r} + 2\Gamma^{\phi}_{rt} + 2\Gamma^{\phi}_{r\phi}\Omega - 2\Gamma^{t}_{rt}\Omega - 2\Gamma^{t}_{r\phi}\Omega^{2})(2\Gamma^{r}_{\phi t} + 2\Gamma^{r}_{\phi\phi}\Omega) \ , \tag{5}$$

where Γs are Christopher connections, which are defined as $\Gamma^{\alpha}_{\beta\gamma} = \frac{1}{2}g^{\alpha\delta}(g_{\beta\delta,\gamma} + g_{\gamma\delta,\beta} - g_{\beta\gamma,\delta})$. The simplified expression of Eq. (5) is(Okazaki et al. 1987)

$$\kappa^{2} = \frac{M(r^{2} - 6Mr + 8aM^{\frac{1}{2}}r^{\frac{1}{2}} - 3a^{2})}{r^{2}(r^{\frac{3}{2}} + aM^{\frac{1}{2}})} \ . \tag{6}$$

Here we only derive the epicyclic frequency referring to the prograde orbits, in which we are most interested. Because there is a maximum of this frequency at a radius outside the marginal stable circular orbit, this means that there would be oscillation modes trapped in the inner region of the accretion disk. This is important for the relativistic precession model of QPOs.

3 RADIAL EPICYCLIC FREQUENCY IN A PERFECT FLUID DISK: SOME MODIFICATIONS

Before we use the radial epicyclic frequency in an accretion disk around compact objects to interpret the observed QPO phenomenon, we need to pay attention to the problems in using the expression of the radial epicyclic frequency obtained in a test particle scenario. So the investigation on the radial epicyclic frequency in a perfect fluid disk is worthwhile.

This has been done in a Schwarzschild case by Rezzolla et al. (2003). However, in a real accretion system, the space-time is usually depicted by a Kerr metric. For example, a neutron star is usually spinning, so the metric portraying the vicinity of the neutron star should be a Kerr metric rather than a Schwarzschild one. Therefore, the extension of the study of Rezzolla et al.(2003) to a Kerr case is necessary.

3.1 Schwarzschild Case

Let's consider a Schwarzschild metric

$$ds^2 = -e^{2\nu(r)}dt^2 + e^{2\lambda(r)}dr^2 + dz^2 + r^2 d\phi^2 \ . \tag{7}$$

We start with a set of conservation equations

$$T^{\alpha\beta}{}_{;\alpha} = 0 \ , \tag{8}$$

where $T^{\alpha\beta} = (e+p)u^\alpha u^\beta + pg^{\alpha\beta}$. An additional equation of the conservation of matter should be also included, that is

$$(\rho u^\alpha)_{;\alpha} = 0 \ . \tag{9}$$

Here the four-velocity is defined as $u^\alpha = u^t(1, 0, \Omega, 0)$. This means that we omit the radial velocity of the fluid. Now we introduce perturbations to the equations above. Since we include the fluid pressure, we need to consider the perturbation of it by defining $\delta Q = \frac{\delta p}{e+p}$. We assume that the perturbations have the form of

$$\begin{pmatrix} \delta v^r \\ \delta v^\phi \\ \delta Q \end{pmatrix} \sim e^{-i\sigma t + ikr} \ . \tag{10}$$

Then we made the following approximations. First, we assume a local approximation $1 \ll kr$, which means the perturbation wavelength is much shorter than the disk radius. The imaginary part of the epicyclic frequency is also ignored, since it has no effect on the real value of epicyclic frequency. This equals to set $k = 0$ in the expression of σ^2, which can be obtained from a dispersion relation of the perturbed equations. The result is (Rezzolla et al. 2003)

$$\kappa^2 = 2e^{-2\lambda}\Omega\left(2\Omega + r\frac{d\Omega}{dr} - 2r\Omega\frac{d\nu}{dr}\right)\left(1 - r\frac{e^{2\nu}\partial_r\nu - \Omega^2 r}{e^{2\nu} - \Omega^2 r^2}\right) \ , \tag{11}$$

where λ and ν follow the definition in Eq.(7), Ω is the angular velocity of the fluid.

3.2 Kerr Case

Following the same steps as in the Schwarzschild case, we can obtain the radial epicyclic frequency of a perfect fluid disk in a Kerr space-time. We list here only the final result without writing the exact expressions of the perturbed equations and dispersion relation, due to their complexity.

In the Kerr space-time, the square of the radial epicyclic frequency can be written as:

$$\kappa^2 = -\left(\frac{\partial\Omega}{\partial r} + 2\Gamma^\phi_{rt} + 2\Gamma^\phi_{r\phi}\Omega - 2\Gamma^t_{rt}\Omega - 2\Gamma^t_{r\phi}\Omega^2\right) \times \left[2\Gamma^r_{\phi t} + 2\Gamma^r_{\phi\phi}\Omega + 2u^{t^2}(g_{\phi\phi}\Omega + g_{t\phi})(\Gamma^r_{tt} + \Omega\Gamma^r_{t\phi}) + 2\Omega u^{t^2}(g_{\phi\phi}\Omega + g_{t\phi})(\Gamma^r_{t\phi} + \Omega\Gamma^r_{\phi\phi})\right] . \tag{12}$$

Using an expression of pressure gradient (Abramowicz, Jaroszyński & Sikora, 1978)

$$\frac{\partial_r p}{e+p} = -\partial_r \ln(u_t) + \frac{\Omega\partial_r l}{1 - \Omega l} \ , \tag{13}$$

we can find

$$2u^{t^2}(g_{\phi\phi}\Omega + g_{t\phi})(\Gamma^r_{tt} + \Omega\Gamma^r_{t\phi}) + 2\Omega u^{t^2}(g_{\phi\phi}\Omega + g_{t\phi})(\Gamma^r_{t\phi} + \Omega\Gamma^r_{\phi\phi}) = -2\frac{\partial_r p}{e+p}g^{rr}(g_{t\phi} + \Omega g_{\phi\phi}) \ . \tag{14}$$

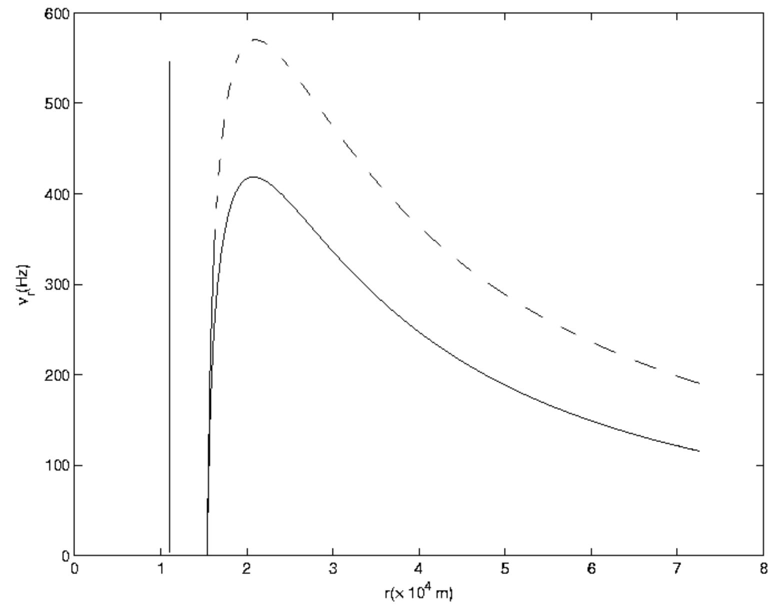

Fig. 1 Comparison of the epicyclic frequency in a test particle case(solid line) and a perfect fluid disk case(dashed line). The x-axis indicates the radial coordination, and the y-axis indicates the frequency of the radial oscillation ($\nu_r = \kappa/2\pi$). For both cases, the mass of the central object is 1.4 $M_\odot$, the spin rate is $a = 0.2$ and the angular velocity is $\Omega = \frac{M^{1/2}}{r^{3/2}+aM^{1/2}}$. The vertical line points out the position of the marginal stable orbit($r = 5.3093r_g$).

So the equivalent form of the epicyclic frequency can be written as

$$\kappa^2 = -(\frac{\partial\Omega}{\partial r} + 2\Gamma^{\phi}_{rt} + 2\Gamma^{\phi}_{r\phi}\Omega - 2\Gamma^{t}_{rt}\Omega - 2\Gamma^{t}_{r\phi}\Omega^2)(2\Gamma^{r}_{\phi t} + 2\Gamma^{r}_{\phi\phi}\Omega - 2\frac{\partial_r p}{e+p}g^{rr}(g_{t\phi} + \Omega g_{\phi\phi}))\ . \quad (15)$$

This expression is consistent with that in two special cases we discussed above. When we set the spin rate of the black hole to zero($a = 0$) in Eq. (12), it equals to Eq. (11). And when we set $\partial_r p = 0$ in Eq. (15), it goes back to the exact form of Eq. (5). Therefore, Eq. (15) can be regarded as a general expression of radial epicyclic frequency in the Kerr space-time for a perfect fluid disk.

4 DISCUSSION

In this paper we derive the radial epicyclic frequency in the perfect fluid disk scenario in Kerr space-time, which is the extension of a previous one in Schwarzschild metric obtained by Rezzolla et al. (2003). This radial epicyclic frequency in the perfect fluid disk scenario also deviates from the test particle case.

As we have mentioned before, the radial epicyclic frequency derived in the test particle case has a maximum at some radius of the disk. However, here we find that when we use some angular momentum models referring to different angular velocities deviating from the Keplerian one (we use the linear angular momentum model following Rezzolla et al. (2003)), there is no such a maximum in a perfect fluid disk, because the function depicting the frequency has only a maximum at a radius smaller than the radius of the marginal stable orbit where an accretion disk can not exist. But in a realistic angular momentum model, the angular velocity should be close to Keplerian. When the Keplerian angular momentum model is used, there are no such problems (See Fig. 1).

The radial epicyclic frequency is one of the most important frequencies which are used to interpret the QPOs in neutron star X-ray binaries (Stella et al. 1999), with a somehow large neutron star mass needed. We find that when we use the epicyclic frequency obtained in a perfect fluid disk, an even larger neutron star mass is needed. We know that a larger neutron star mass refers to a relatively smaller epicyclic frequency. For the same neutron star mass, the epicyclic frequency in a perfect fluid disk case is larger than the one in a test particle case (See Fig. 1). If we use the linear angular momentum rather than the Keplerian one, the lack of maximum of the epicyclic frequency may imply some more difficulties in using the relativistic precession model. If there is no maximum epicyclic frequency, there would not be oscillation mode trapped in the inner region of the disk. However, there may be also an alternative possibility. For example, there could be something wrong in the method we use to estimate the innermost radius of the accretion disk (Watarai & Mineshige 2003).

We should notice that in the two situations (a test particle case and a perfect fluid disk case), we both neglect an important factor: accretion or simply the radial velocity of the test particle or the fluid. So the deviation of the fitted neutron star mass using both frequencies may come from the omission of the radial component of motion of test particle or fluid.

In our future work, we will look into the problem of how to rationally decide the innermost radius of the accretion disk, and also how the accretion influences the expression of the radial epicyclic frequency in both a test particle case and a perfect fluid disk case.

Acknowledgements We thank all members in our group at Peking University. The discussion with them helped us to understand some key points in deriving the epicyclic frequency. The work was partly supported by the NSFC (No. 10173001).

References

Abramowicz M., Jaroszyński M., Sikora M., 1978, A&A, 63, 221
Kluźniak W., Abramowicz M. A., Kato S. et al., 2004, ApJ, 603, L89
Okazaki A. T., Kato S., Fukue J., 1987, PASJ, 39, 457
Rezzolla L., Yoshida S., Zanotti O., 2003, MNRAS, 344, 978
Stella L., Vietri M., 1999, Phys. Rev. Lett.82, 17
Van Der Klis M., 2000, ARA&A, 38, 717
Watarai K., Mineshige S., 2003, PASJ, 55, 959