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Statistical Properties of the Combined Emission of a Discrete Source Population: Astrophysical Implications

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Abstract We study the statistical properties of the combined emission of a population of discrete sources. In particular, we consider the dependence of their total luminosity ($L_{\rm tot} = \sum_k L_k$) and fractional $\rm rms_{tot}$ variability on the number of sources n or on the normalization of the luminosity function. We show that due to small number statistics a regime exists, in which L_{tot} grows non-linearly with n. This is in apparent contradiction with the seemingly obvious prediction $\langle L_{\rm tot}\rangle = \int L dN/dL dL \sim n$. In this non-linear regime, the $\rm rms_{tot}$ decreases with n significantly more slowly than expected from the rms$\sim 1/\sqrt{n}$ averaging law. Only in the limit of $n \gg 1$ do these quantities behave as intuitively expected, $L_{\rm tot} \sim n$ and $rms_{\rm tot} \sim 1/\sqrt{n}$. Using the total X-ray luminosity of a galaxy due to its X-ray binary population as an example, we show that the $L_{\rm X}$–SFR and $L_{\rm X}$–M$_*$ relations predicted from the respective "universal" luminosity functions of high and low mass X-ray binaries are in a good agreement with observations.

Key words: Statistics – X-ray binaries

1 BASIC STATISTICAL PROPERTIES

The statistical properties of a power law distribution are interesting because combined quantities from summing over a power law distribution show, at first glance, counterintuitive behavior. In particular the total X-ray luminosity versus star formation rate (SFR) exhibits a more complex behavior. This relation, as shown in the left hand panel of Figure 5, is non-linear at low star formation rate/X-ray luminosities, and the relation becomes linear only at high star formation rates. As shown below this behavior results naturally from the properties of the luminosity function of X-ray binaries. In the following we will discuss the interesting statistical properties of the power law probability distribution using the X-ray binary luminosity function as an example. Note, however, that the statistical properties hold for any power law distributed random variable.

As Grimm et al. (2003) have shown the normalization of the HMXB luminosity function is directly proportional to the star formation rate in a galaxy. Thus the straightforward way to compute the total X-ray luminosity of a galaxy due to HMXBs from the luminosity function is simple integration, i.e. the expectation value of the total luminosity $L_{\rm tot}$:

$$\langle L_{\rm tot}\rangle = \int_{L_1}^{L_2} L N_0 L^{-\alpha}, \qquad (1)$$

with N_0 the normalization and α the (differential) slope of the power law luminosity function, and L_1 and L_2 the lower and upper limits of the validity of the power law distribution. The existence of a lower

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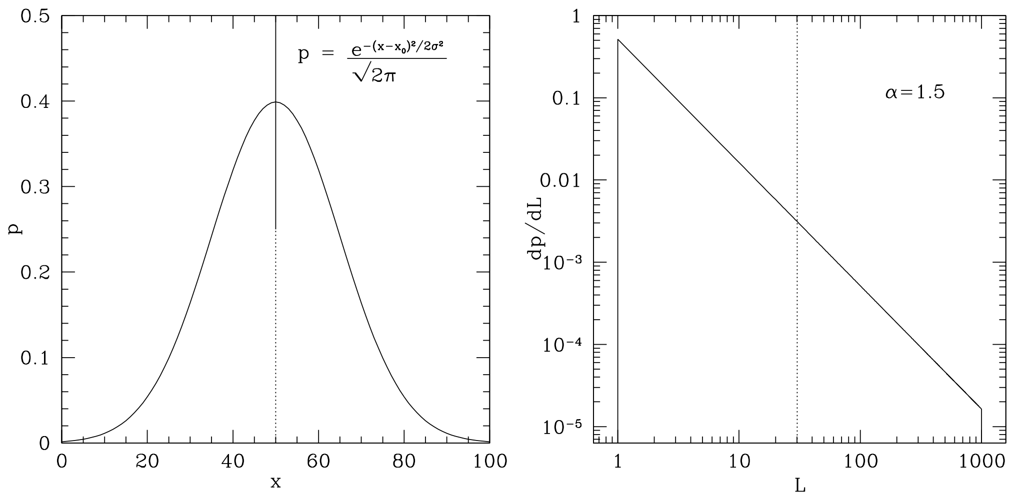

Fig. 1 *Left:* Gaussian probability distribution with a mean of 50 and a standard deviation of 15. *Right:* A power law distribution from 1 to 1000 and a differential slope of 1.5. The dotted lines indicate the mean of the distributions. On the left side the solid line indicates the mode as well. The mode of the power law is always at the lower end of the distribution.

limit derives simply from having a finite number of sources. The existence of an upper limit might be due to physical constraints, that will be discussed below. The quantity in Eq. 1 is obviously always directly proportional to the normalization.

The difference between this simple estimate and the observational results arises because the luminosity function is a probability distribution of a discrete number of sources. That the probability distribution has the form of a power law leads to the important distinction between the mode of the distribution, the most likely value to observe, and the mean of the distribution, the average value of the distribution. For the often-used Gaussian distribution this distinction disappears as is illustrated on the left hand side in Figure 1.

The left hand side of the figure shows a Gaussian probability distribution with a mean of 50 and a standard deviation of 15. The dotted line indicates the mean of the distribution and the solid line the mode of the distribution. The two are identical for a Gaussian distribution. The right hand side of Figure 1 shows a power law distribution from 1 to 1000 and a differential slope of 1.5. The dotted line again indicates the mean of the distribution. The mode of a power law distribution is obviously always at the lower boundary of the distribution, at 1 in this case. Note that this is only the case for a single source.

One of the properties of a Gaussian distribution is that the sum of Gaussian distributed random variables follows a Gaussian as well. However, as will be shown below the sum of power law distributed variables is not a simple power law. The mode of the power law distribution for more than 1 source changes from the lower limit of the validity range to the expectation value of Eq. 1 with increasing number of sources. For a sufficient number of sources the Central Limit Theorem applies and the distribution becomes Gaussian.

Another property of a luminosity function is that it consists of discrete sources. The distribution is not continuous. An important consequence of this is that for low normalizations of the luminosity function a random sampling of the distribution will most likely result in a brightest source whose luminosity does *not* equal the upper validity limit of the luminosity function. The point is illustrated in the left hand side of Figure 2. Although the luminosity function extends from 1 to 1000 by definition, the normalization is too small for the brightest source to reach a luminosity of 1000.

An obvious consequence of the discrete nature of the luminosity function distribution is that the most likely luminosity of the brightest source depends on the normalization up to the point when the normalization is large enough that a sufficient number of sources are close to the maximum luminosity. Because for small enough slopes of the luminosity function the total luminosity is dominated by the brightest source(s),

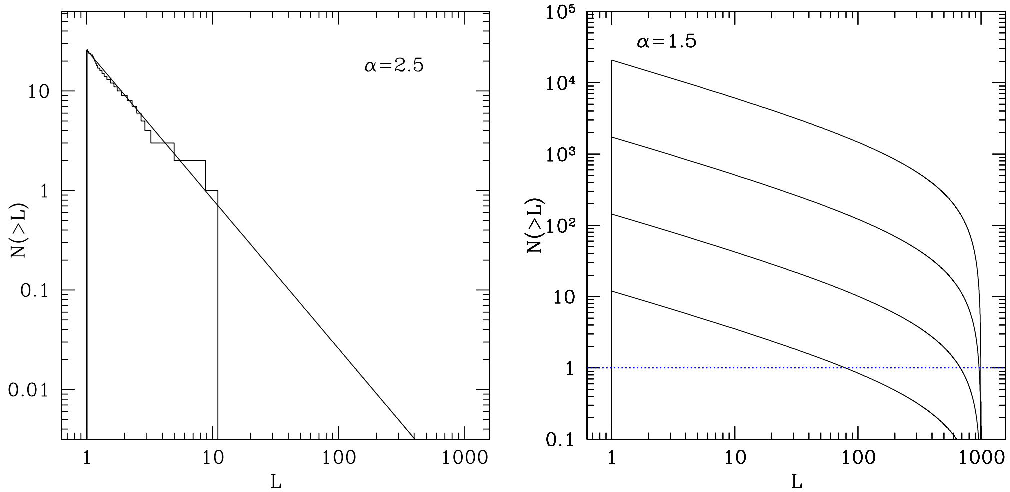

Fig. 2 *Left:* Cumulative power law luminosity function (solid line) with a sampling of discrete sources (histogram). *Right:* Schematic cumulative power law luminosity function for various normalizations. Note the changes of normalization *and* luminosity of the brightest source from the lowest to the highest normalization. For clarity the luminosity functions extend to (unphysical) numbers of less than one.

this leads to the non-linear behavior in relations of total X-ray luminosity versus normalization. To compute the expected total luminosity for a given luminosity function L_2 in Eq. 1, the upper limit of the luminosity function, has to be replaced by the most likely luminosity of the brightest source $L_{\max}$. This itself is dependent on the normalization N_0. Thus the total luminosity is dependent on the normalization not only through the normalization as a multiplier in Eq.1, but also through $L_{\max}(N_0)$. This holds until $L_{\max}(N_0) \geq L_2$. The right hand side of Figure 2 illustrates schematically the change of the cumulative luminosity function with increasing normalization.

2 COMPUTATION

The property of interest is the probability distribution of the total X-ray luminosity from a number of n sources. The probability distribution for each individual source is a power law with a (differential) slope of α in a luminosity range from L_1 to L_2.

A computationally convenient way to compute the probability distribution is to quantify it as the product of the Poissonian distribution of n sources with the probability distribution of the total luminosity of exactly n sources.

$$p(L_{\rm tot}) = \sum_{n=0}^{n=\infty} P_\mu(n) p_n(L_{\rm tot}) \,. \tag{2}$$

Assuming $\langle n \rangle \gg 1$ the Poisson distribution can be approximated by the delta function and thus

$$p(L_{\rm tot}) \approx p_n(L_{\rm tot}). \tag{3}$$

Using the convolution theorem the probability distribution for the total luminosity can be expressed by the probability distribution for one source

$$\hat{p_n} = \hat{p_1}^n, \tag{4}$$

with

$$\hat{p_1} = \int_0^\infty p_1(L) e^{i\omega L} dL. \tag{5}$$

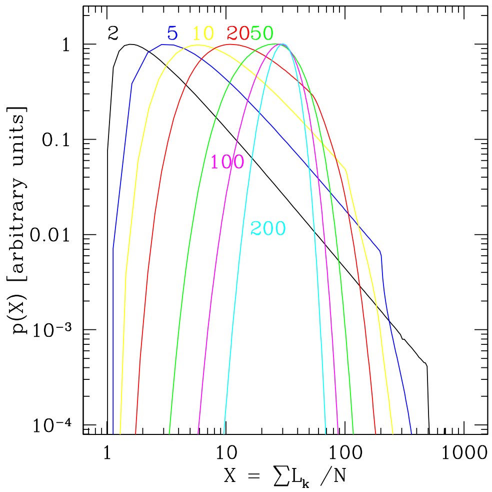

Fig. 3 Probability distributions of the average luminosity $\sum_{k=1}^{n} L_k/n$ of n discrete sources with a power law with slope 1.5 and lower and upper luminosity cutoffs of $L_1 = 1$ and $L_2 = 1\,000$.

The probability distribution of the total X-ray luminosity therefore becomes

$$p_n(L_{\rm tot}) = \int_{-\infty}^{\infty} \hat{p_n}(\omega) e^{-i\omega L_{\rm tot}} d\omega \,, \tag{6}$$

where the hatted symbols are the Fourier transforms of the respective probability distribution. The resulting probability distributions for various numbers of sources are shown in Figure 3.

For more details on the computation please refer to Gilfanov et al. (2004).

3 EXAMPLES

The behavior of the luminosity function with increasing normalization has various important implications for the study of integrated properties derived from the luminosity function. The following examples use mostly X-ray luminosity as the quantity of interest. But the statistical behavior is the same for any kind of power law distribution.

The following relies on the assumption that the power law representation of the luminosity function does not change, apart from long-term trends in its normalization. However, it is well known that X-ray binaries are strongly variable. Persistent sources show variability of more than an order of magnitude in luminosity. And transient sources are only observable for a relatively short time after which they disappear completely. This raises the question how valid the assumption of a constant luminosity function is. For the Milky Way Grimm et al. (2006) use several hundred luminosity functions obtained with ASM to show that the variability of the shape of the luminosity function is quite small. Figure 4 shows the HMXB luminosity functions for 1 513 days. Also in other galaxies Chandra observations have shown no significant variability in the luminosity function of different galaxies like M33 Grimm et al. (2005) and the Antennae Zezas et al. (2004). This leads to the conclusion that Chandra or XMM luminosity functions are a good representation of the underlying X-ray source population, and the stability supports the conclusions we draw from our statistical arguments.

3.1 Measure of Star Formation and Mass

Assuming universal luminosity functions for HMXBs and LMXBs, Grimm et al. (2003) and Gilfanov (2004) make predictions for the expected X-ray luminosity from HMXBs or LMXBs given a star formation rate or a stellar mass. The predictions with data are shown in Figure 5. On the left hand side X-ray luminosity versus star formation rate is shown. The continuous line is the prediction obtained

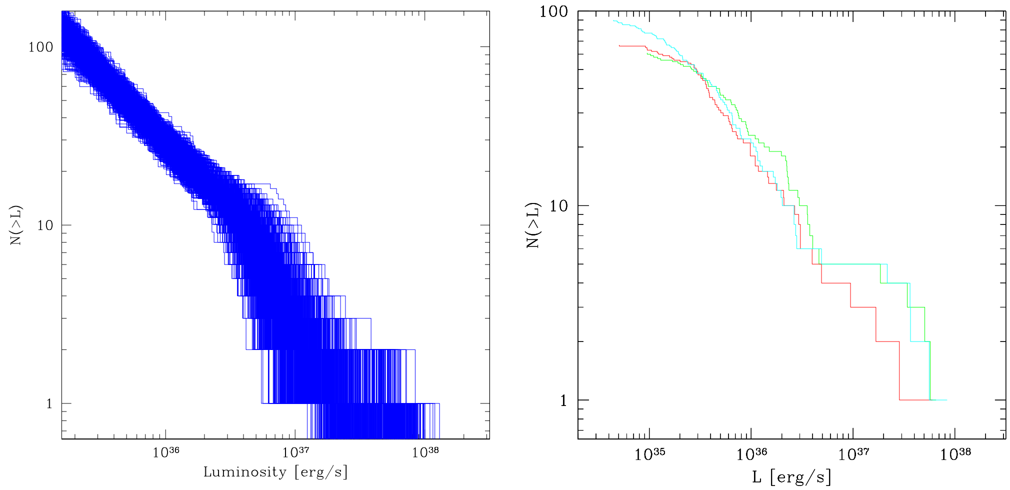

Fig. 4 *Left:* 2–10 keV HMXB luminosity functions for 1 513 days. *Right:* Luminosity functions of M33 for 3 different observations.

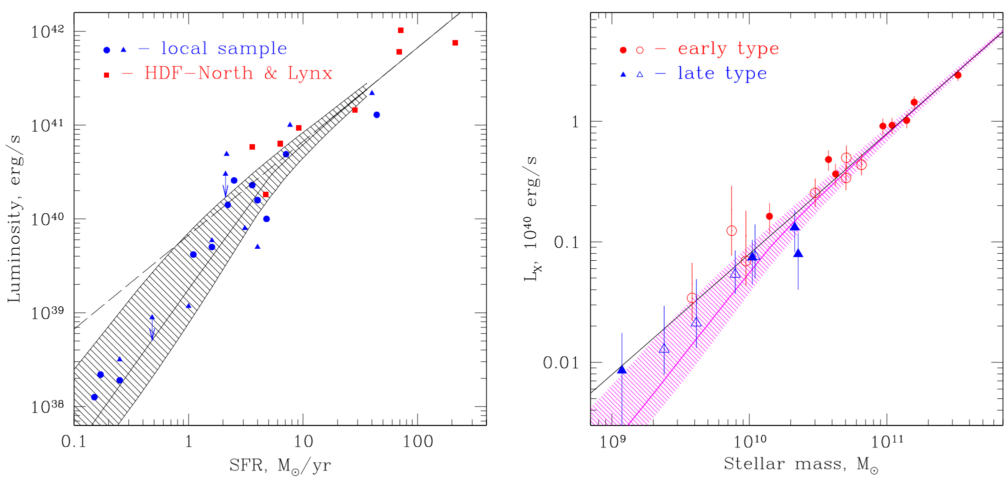

Fig. 5 *Left:* Total 2–10 keV X-ray luminosity versus star formation rate for samples of local galaxies and galaxies in the Hubble Deep Field. *Right:* Total X-ray luminosity versus stellar mass for samples of local galaxies from Gilfanov (2004).

from the universal luminosity function for HMXBs, and the shaded area corresponds to the 68% CL area. Note that this is asymmetric at low star formation rate due to the tail of the probability distribution to high luminosities. As can be seen there is a break around SFRs of around 7 $M_\odot$ yr^{-1}, or X-ray luminosities around $2-3\times 10^{40}$ erg s^{-1}. On the right hand side the relation between total X-ray luminosity and stellar mass of a galaxy from Gilfanov (2004) is plotted. The continuous line is the prediction from the universal luminosity function of LMXBs. The shaded area is again the 68% CL area.

3.2 Cutoff in HMXB Luminosity Function

The left hand side of Figure 5 shows a break in the relation between total X-ray luminosity and star formation rate. For higher luminosities and/or star formation rates the relation becomes linear. However, this relation can only become linear, if the luminosity of the brightest source does not grow beyond a certain

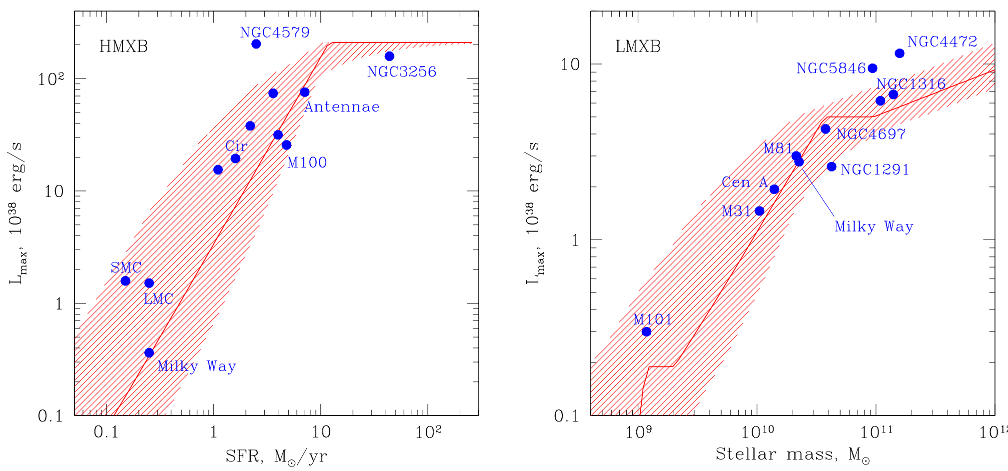

Fig. 6 0.3–8 keV X-ray luminosity of the brightest source for HMXBs and LMXBs versus SFR and mass.

limit. Therefore the break implies that the luminosity function of HMXBs has a cutoff. Note that this evidence is independent of our knowledge of the shape of the luminosity function.

3.3 Brightest Source as SFR/Mass Indicator

The existence of a non-linear regime in the X-ray luminosity–SFR relation implies a dependence of the luminosity of the brightest source on the normalization of the luminosity function. Thus as long as the non-linear relation holds the brightest source alone is an indicator of star formation rate or mass. The prediction from the universal luminosity functions and observations are shown in Figure 6.

3.4 Aperiodic Variability

As a result of the non-linear behavior aperiodic variability, measured in rms, of a population of sources in the low number regime does not follow the usual $1/\sqrt{n}$ rule. Assuming for simplicity that all sources have the same rms value rms_0 and depending on the slope of the luminosity function the total rms value is constant if the number of sources is small enough. In Figure 7 the predictions for HMXBs and LMXBs, obtained from the universal luminosity functions, are shown.

3.5 Nature of ULXs

One of the most interesting question at present concerns the nature of the so-called ultraluminous X-ray sources, defined as sources with an apparent isotropic luminosity larger than $\sim 10^{39}$ erg s^{-1}. One possible explanation is that these sources are intermediate mass black holes with masses between 100 and 10 000 solar masses. If this is the case these sources would most likely have a different luminosity function than normal X-ray binaries. Since ULXs are very rare, the normalization of this LF would be very small. However, normal X-ray binaries have a cutoff in the luminosity function. The extension of the ULX luminosity function would lead to another break in the total X-ray luminosity–SFR relation that would become non-linear again, as shown in Figure 8. If any deviation from linearity to high SFR can be detected this would be an indication for the existence of intermediate mass black holes.

3.6 Parameters of the Universal Luminosity Functions

As mentioned before, the non-linear behavior of the total X-ray luminosity depends on the shape of the luminosity function. One can invert this relation, and deduce parameters of the luminosity function from observations of unresolved galaxies. In particular, the slope β in the non-linear regime depends only on the slope α of the luminosity function

$$\beta = \frac{1}{\alpha - 1} \tag{7}$$

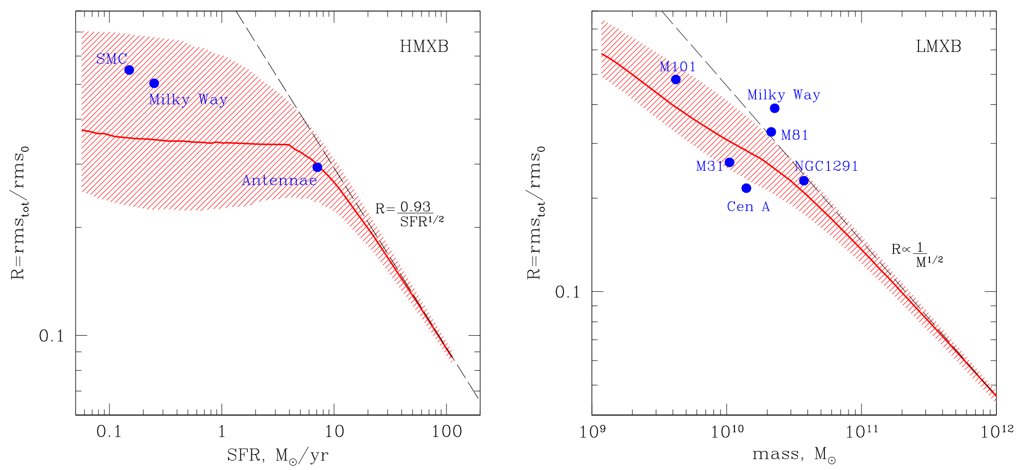

Fig. 7 Aperiodic variability for HMXBs and LMXBs versus SFR and stellar mass, respectively.

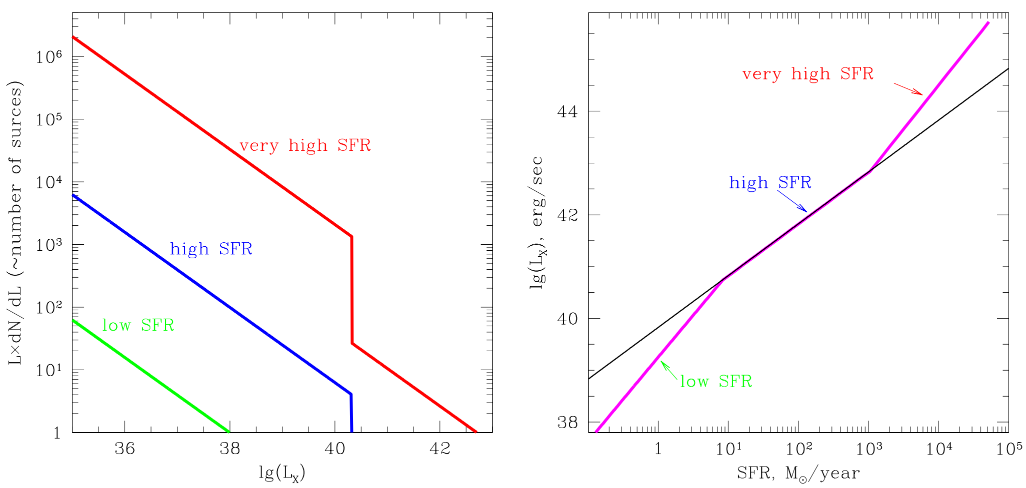

Fig. 8 Sketch of luminosity function and total X-ray luminosity versus SFR for hypothetical intermediate mass black holes.

and the break luminosity is dependent on the slope α and the cutoff luminosity. In the case of $\alpha > 2$ it depends also on the lower luminosity cutoff because in this case emission from low luminosity sources dominates the total luminosity

$$L_{\text{break,tot}} \approx \begin{cases} \frac{L_2}{2-\alpha} & \text{for } 1 < \alpha < 2, \\ \frac{L_2}{2-\alpha} \times \left(\frac{L_2}{L_1}\right)^{\alpha-2} & \text{for } \alpha > 2\,. \end{cases} \tag{8}$$

Sampling the total luminosity range sufficiently, slope and cutoff of the luminosity function can be determined independently of the knowledge of the luminosity function. Moreover, if there are second-order effects affecting the shape of the luminosity function a large sample of galaxies may make these effect visible as systematic scatter in the simple luminosity relations.

4 CONCLUSIONS

The study of the statistical properties of a power law probability distribution leads to various tools for investigating the properties of a discrete source population and the host galaxy of the population. In the case of X-ray binaries we have shown that the behavior of integrated properties, and even individual sources, can be used to relate X-ray luminosity and rms to the star formation rate or the stellar mass of a galaxy. Moreover, due to statistical effects the integrated properties contain information about the underlying luminosity function even if individual objects are not observable. This can even shed new light on the nature of the objects in the luminosity function.

It is important to note that the statistical results rest only on the condition that mode and mean of the probability distribution are different for an individual source and that the objects of interest are discrete sources. Therefore the above analysis is in principle applicable to a wider variety of situations, and not only to X-ray binaries as in the example here.

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DISCUSSION

J. Beckman: Your statistical deduction and illustration are instructive and illuminating. The problem with the plot you showed is not with the X-ray data but with the SFR estimate. They may be model dependent, and luminosity dependent. This is because a fraction of the photons may escape the galaxy and escape detection, like in the FIR.

H.-J. GRIMM: I agree that the SFR measurements are the most uncertain part of the X-ray luminosity–SFR relation. The best one can do at the moment is to take UV and FIR fluxes together to estimate SFR as this should address the problem of unknown escape fraction.

J. Beall: What will beaming of sources do to the luminosity function?

H.-J. GRIMM: Most likely beaming will not significantly affect the luminosity function. If only a fraction of sources is beamed, the ordering of sources is changed but not the overall shape of the LF. In the pathological case of all sources being beamed into the line of sight, the luminosity of all sources would increase. This would appear as an increase in normalization. Körding, Markoff and Falcke were able to reproduce the observed LF with their jet model, assuming all X-rays originate in the jet.